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Abstract. It is demonstrated that a compact spinor space–time manifold need not be parallelisable. Necessary and sufficient conditions are derived for a parallelisation and counter examples are constructed.

1. Introduction

Most smooth four-manifolds used in general relativity are assumed to be non-compact. This is theoretically necessary if causality is required because in certain circumstances, compact space–times can have closed time-like curves. However, compact space–times can provide useful and interesting examples of Lorentzian four-manifolds. In order to define global spinor fields on space–times, one must be able to replace the proper orthochronous Lorentz group by its universal covering group SL(2, ℂ). Such a requirement places a stringent topological condition upon the underlying four-manifold. Geroch (1968) demonstrated that a non-compact space–time can admit a spinor structure if and only if it is parallelisable. Lee (1975) claimed that the theorem also applies to compact space–times. It is the purpose of this paper to demonstrate that Geroch's theorem does not in fact extend to compact space–times. A necessary and sufficient condition is obtained for a compact spinor space–time to be parallelisable, and in so doing, a family of counter examples is constructed.

2. Topological machinery

Suppose that G is a Lie group. Then there exists a universal classifying space BG for principal G-bundles over paracompact base spaces. BG is the base space of the universal G-bundle ξ_G. The bundle is universal in the sense that if ξ is any principal G-bundle over a paracompact base X, there exists a map $f: X \to BG$ (defined up to a homotopy) such that ξ is bundle isomorphic to the pull-back (Husemoller 1966), $f^*(\xi_G)$ of ξ_G from BG to X along f. A principal G-bundle can admit a cross section if and only if it is bundle isomorphic to the trivial principal G-bundle on its base, the topological product of G by the base. Equivalently, a principal G-bundle can admit a cross section if and only if its classifying map is homotopic to a map which lifts from BG to the total space EG of ξ_G. One can approach the problem of deciding when such liftings exist by examining the problem of finding a cross section (or lifting of the identity map of BG) of ξ_G. The algebraic obstructions to lifting the identity map of

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BG are the universal cohomology obstruction classes:

$$\theta_K \in H^k(BG, \Pi_{k-1}(G)).$$

Such a lifting exists if and only if all the obstruction classes are trivial. Because G is a Lie group $\Pi_1(G)$ and therefore $\Pi_k(G)$ for all k are Abelian groups. If A is an Abelian group, the generators of the cohomology algebra $H^*(BG, A)$ are called universal characteristic classes in A for principal G-bundles. By definition, the universal obstruction classes can be expressed in terms of the universal characteristic classes. The former are universal in the sense that if $f^*: H^*(BG, A) \to H^*(X, A)$ is the cohomology homomorphism induced by the classifying map f of a principal G-bundle $\xi \cong f^*(\xi_G)$ with base X, then the classes $f^*(\theta_k)$ are the obstructions to a cross section of ξ. Clearly, $f^*(\theta_k)$ are expressible in terms of the pulled-back universal characteristic classes, the characteristic classes of ξ. We shall need to consider the following cases.

(1) $G = \mathrm{O}(n)$, $A = \mathbb{Z}_2$. The cohomology algebra $H^*(\mathrm{BO}(n), \mathbb{Z}_2)$ is generated over $\mathbb{Z}_2$ by universal Stiefel–Whitney characteristic classes $\langle W_i \rangle$, $i = 0, n$, where $W_i \in H^i(\mathrm{BO}(n), \mathbb{Z}_2)$ and $W_0 = 1$. Note that the monomials $\Pi_{i=1}^m W_{q_i}^{m_i}$ with $\Sigma_{i=1}^m m_i \,.\, q_i = q$ form bases for $H^q(\mathrm{BO}(n), \mathbb{Z}_2)$ over $\mathbb{Z}_2$.

(2) $G = \mathrm{SO}(n)$, $A = \mathbb{Z}_2$. The cohomology algebra $H^*(\mathrm{BSO}(n), \mathbb{Z}_2)$ is isomorphic to the quotient of $H^*(\mathrm{BO}(n), \mathbb{Z}_2)$ by the ideal generated by W_1. This follows from the group isomorphism $\mathrm{SO}(n) \cong \mathrm{O}(n)/\mathbb{Z}_2$ and the fact that $H^*(\mathrm{B}\mathbb{Z}_2, \mathbb{Z}_2)$ is the polynomial algebra on W_1. Note in particular that $H^1(\mathrm{BSO}(n), \mathbb{Z}_2) = 0$ and that $H^2(\mathrm{BSO}(n), \mathbb{Z}_2)$ is generated by W_2.

(3) $G = \mathrm{SO}(n)$, $A = \mathbb{Z}$. Let ${}_2I$ be the ideal of elements of order two in the cohomology algebra $H^*(\mathrm{BSO}(n), \mathbb{Z})$. Then the algebra $H^*(\mathrm{BSO}(n), \mathbb{Z})/{}_2I$ is freely generated over $\mathbb{Z}$ by universal Pontryagin classes $\langle P_i \rangle$, $i = 0, n/2$, where $P_0 = 1$ and $P_i \in H^{4i}(\mathrm{BSO}(n), \mathbb{Z})$, with the additional generator E the universal Euler class in $H^n(\mathrm{BSO}(n), \mathbb{Z})$ if n is even. Note that $n/2 \equiv K$ if $n = 2K$ or $n = 2K + 1$.

In the applications, much use will be made of the rich algebraic structure of cohomology algebras. The following structures and invariants will all be required.

(4) If X is compact and orientable over A, the top dimensional homology group $H_n(X, A)$ is isomorphic to A. (Note that if X is non-compact and orientable over $\mathbb{Z}$, $H_n(X, A) = 0$.) The generator of $H_n(X, A)$ is called the fundamental class of X in A and will be denoted by $\langle X \rangle$.

(5) Homology and cohomology with coefficients in A are paired to A via the Kronecker product ($\otimes$ denotes the usual tensor product). This is the bilinear map:

$$H^k(X, A) \otimes H_k(X, A) \to A$$

$$v \otimes x \mapsto v \,.\, x.$$

If $p \in H^n(X, A)$ is a linear combination of n-dimensional products of characteristic classes in A, the Kronecker product $p \,.\, \langle X \rangle$ is called an A-characteristic number. It is a classical result of cobordism theory that the Pontryagin and Stiefel–Whitney numbers are oriented cobordism invariants.

(6) The topological signature or index of an oriented $4k$-manifold is a particularly

useful tool. It is defined as follows. The cohomology cup products (Spanier 1966)

$$.\cup.: H^{2k}(X, \mathbb{R} \otimes H^{2k}(X, \mathbb{R}) \to H^{4k}(X, \mathbb{R})$$

$$.\cup.: x \otimes y \mapsto x \cup y$$

together with the Kronecker product with the fundamental class defines a non-degenerate symmetric bilinear form $q_X(x, y) = (x \cup y) . \langle X \rangle$ on the real linear space $H^{2k}(X, \mathbb{R})$. The signature of X, sig(X), is defined as the signature of q_X. This integer is another oriented cobordism invariant and is related to the Pontryagin characteristic classes through the Hirzebruch (1966) signature theorem. The Hirzebruch L-genus is a rational polynomial in Pontryagin classes and the signature theorem yields the result that $L_k . \langle X \rangle = \text{sig}(X)$, where L_k is the projection of L in the top dimensional cohomology group. For four-manifolds, this reduces to the theorem of Thom: $\text{sig}(X) = \frac{1}{3}P_1 . \langle X \rangle$. Similarly, the integrality of the rational A-genus when X is a spin manifold yields $A . \langle X \rangle = P . \langle X \rangle / 24$ is an even integer if X is a four-manifold. Therefore, if X is a four-dimensional spin manifold, the integer $P_1 . \langle X \rangle$ is divisible by 48, which is Rohlin's theorem.

(7) The last invariant that we shall need is the Wu class of a four-manifold. Replacing the field $\mathbb{R}$ by $\mathbb{Z}_2$ in (6) yields a linear functional W on $H^2(X, \mathbb{Z}_2)$ defined by $W(x) = x \cup x . \langle X \rangle$. (Note that $(x+y)^2 = x^2 + y^2$ in $\mathbb{Z}_2$ arithmetic.) Therefore, W may be identified with an element of the vector dual space $H_2(X, \mathbb{Z}_2)$ of $H^2(X, \mathbb{Z}_2)$. But recall that the Poincaré duality theorem yields an isomorphism:

$$.\cap \langle X \rangle: \mathrm{H}^2(\mathrm{X}, \mathbb{Z}_2) \to \mathrm{H}_2(\mathrm{X}, \mathbb{Z}_2)$$

$$.\cap \langle X \rangle: x \mapsto x \cap \langle X \rangle$$

where '$\cap$' denotes the bilinear cap-product from cohomology$\otimes$homology into homology (Spanier 1966). There must therefore be a cohomology class V in $H^2(X, \mathbb{Z}_2)$ such that $W(x) = x . (V \cap \langle X \rangle) = (V \cup x) . \langle X \rangle$. This implies that $x \cup x = V \cup x$ for all x, where V is called the Wu class of X. In general, V is a polynomial of dimension $2k$ in the Stiefel–Whitney classes; in our case, $V = W_2$.

3. Spinor space–times

In this section, we shall re-derive the theorem of Hirzebruch and Hopf (1968) which states necessary and sufficient conditions for a given compact orientable four-manifold to be parallelisable. We specialise these results to the case of a compact spinor space–time. In our constructions, much use will be made of the following. If X is a compact oriented four-manifold, a spherical modification of X is a manifold X_s obtained by excising the interior $\mathring{D}^4$ of a closed four-disc D^4 from X and from $S^1 \times S^3$ and identifying the resulting boundary three-spheres. The following theorem summarises the properties of spherical modification that we shall use below.

Theorem 1. Let X be a compact orientable four-manifold. Then if X_s is a spherical modification of X, the characteristic classes are related as follows:

(1) $W_2(X) \neq 0$ if and only if $W_2(X_s) \neq 0$

(2) $\mathrm{E} . \langle X_s \rangle = \mathrm{E} . \langle X \rangle - 2$

(3) $P_1 . \langle X_s \rangle = P_1 . \langle X \rangle$

Proof. (1) W_2 is identified with the Wu class and we therefore use the fact that $W_2 \neq 0$ if and only if there exists a mod(2) cohomology class x with $x \cup x \neq 0$. Firstly, $W_2(X) \neq 0$ implies that $W_2(X_s) \neq 0$. Let D^4 be a closed four-disc in X and let $X' = X \backslash \mathring{D}^4$. The inclusion maps $X' \subsetneq X$, $X' \subsetneq X_s$ and $(S^1 \times S^3)' \subsetneq X_s$ are denoted by i, j and k respectively. We assert that the induced cohomology homomorphisms

$$i_2^*: H^2(X, \mathbb{Z}_2) \to H^2(X', \mathbb{Z}_2) \qquad \text{and} \qquad j_2^*: H^2(X_s, \mathbb{Z}_2) \to H^2(X', \mathbb{Z}_2)$$

are isomorphisms. To see this, consider the Mayer–Vietoris sequence:

$$\to H^1(S^3, \mathbb{Z}_2) \underset{d^*}{\to} H^2(X_s, \mathbb{Z}_2) \underset{J_2^*}{\to} H^2(X', \mathbb{Z}_2) \oplus H^2((S^1 \times S^3)', \mathbb{Z}_2) \to H^2(S^3, \mathbb{Z}_2).$$

J_2^* is the homomorphism defined by $J_2^*: x \mapsto j_2^*(x) + k_2^*(x)$. Because $H^1(S^3, \mathbb{Z}_2)$ and $H^2(S^3, \mathbb{Z}_2)$ are trivial groups, J_2^* is an isomorphism by exactness. But then j_2^* is an isomorphism because $H^2((S^1 \times S^3)', \mathbb{Z}_2) = 0$ follows from the Mayer–Vietoris sequence for $S^1 \times S^3 = (S^1 \times S^3)' \cup D^4$ together with the fact that $H^2(S^1 \times S^3, \mathbb{Z}_2) = 0$. Similarly the assertion that i_2^* is an isomorphism follows from the Mayer-Vietoris sequence for $X = X' \cup D^4$. Since $W_2(X) \neq 0$, there exists a class $v \in H^2(X, \mathbb{Z}_2)$ with $v \cup v \neq 0$. Let $v' = i_2^*(v)$, then $v' \cup v' \neq 0$. For, $v' \cup v' = i_2^*(v) \cup i_2^*(v) = i_4^*(v \cup v)$ where i_4^* is the four-dimensional cohomology homomorphism induced by i. Since $v \cup v \neq 0$, $v' \cup v' \neq 0$ will follow if i_4^* is a monomorphism. But this is the case as can be seen by examining the relevant part of the Mayer–Vietoris sequence for $X = X' \cup D^4$ and noting that $H^4(D^4, \mathbb{Z}_2) = 0$ implies that $I_4^* = i_4^*$ and therefore exactness yields $\text{Ker}(i_4^*) = \text{Im}(d_4^*)$ which is a trivial group because S^3 bounds in X. If $v'' = j_2^{*-1}(v') \in H^2(X_s, \mathbb{Z}_2)$, $v'' \cup v'' \neq 0$ because $j_4^*(v'' \cup v'') = j_2^*(v'') \cup j_2^*(v'') = v' \cup v' \neq 0$. Therefore $W_2(X_s) \neq 0$. Conversely, $W_2(X_s) \neq 0$ implies $W_2(X) \neq 0$. Let w be a class in $H^2(X_s, \mathbb{Z}_2)$ with $w \cup w \neq 0$ and let $w' = j_2^*(w)$. Then $w' \cup w' \neq 0$ because $w' \cup w' = j_2^*(w) \cup j_2^*(w) = j_4^*(w \cup w)$ and j_4^* is a monomorphism. The latter assertion follows from the Mayer–Vietoris sequence:

$$H^3(S^3, \mathbb{Z}_2) \underset{d_4^*}{\to} H^4(X_s, \mathbb{Z}_2) \underset{J_4^*}{\to} H^4(X', \mathbb{Z}_2) \oplus H^4((S^1 \times S^3)', \mathbb{Z}_2)$$

$J_4^*(w \cup w) = j_4^*(w \cup w) + k_4^*(w \cup w)$. The class $k_4^*(w \cup w)$ is just $k_2^*(w) \cup k_2^*(w)$ which is trivial because $H^2((S^1 \times S^3)', \mathbb{Z}_2)$ is a trivial group. Therefore $j_4^*(w \cup w) = J_4^*(w \cup w)$ which is non-zero because $\text{Im}(d_4^*) = \text{Ker}(J_4^*) = 0$, because S^3 bounds in X_s. Finally, if we define $w'' = i_2^{*-1}(w')$ in $H^2(X, \mathbb{Z}_2)$, $w'' \cup w'' \neq 0$ because $i_4^*(w'' \cup w'') = w' \cup w' \neq 0$. Therefore $W_2(X) \neq 0$.

(2) $\text{E} . \langle X_s \rangle = \text{E} . \langle X \rangle - 2$ follows easily from the Mayer–Vietoris sequences of the decompositions $X_s = X' \cup (S^1 \times S^3)'$, $X = X' \cup D^4$ and $S^1 \times S^3 = (S^1 \times S^3)' \cup D^4$.

(3) $P_1 . \langle X_s \rangle = P_1 . \langle X \rangle$ follows from the classic result that manifolds related by general spherical modifications are cobordant.

Recall that a parallelisation of a four-manifold X is a cross section of its principal SO(4) frame bundle and that such a section can exist if and only if the pulled-back images of the universal obstruction classes vanish. We need only consider the cases in dimensions one to four because the cohomology of X is trivial in higher dimensions. It is easy to show that the group SO(4) is diffeomorphic to the product $\text{SO}(3) \times S^3$ which is in turn diffeomorphic to $\mathbb{R}P^3 \times S^3$. The homotopy groups of SO(4) are therefore given by

$$\Pi_0(\text{SO}(4)) = \mathbb{Z}, \qquad \Pi_1(\text{SO}(4)) = \mathbb{Z}_2,$$
$$\Pi_2(\text{SO}(4)) = 0 \quad \text{and} \quad \Pi_3 = (\text{SO}(4)) = \mathbb{Z}\oplus\mathbb{Z}.$$

We examine the universal obstructions in turn (clearly $\theta_3 = 0$).

(1) $\theta_1 \in H^1(\text{BSO}(4), \mathbb{Z})$. Recall that $H^1(\text{BSO}(4), \mathbb{Z})$ consists entirely of elements of order two. This implies that the group must be isomorphic to a subgroup of $H^1(X, \mathbb{Z}_2)$, which as we noted above is a trivial group. Therefore $\theta_1 = 0$.

(2) $\theta_2 \in H^2(\text{BSO}(4), \mathbb{Z}_2)$. The space $H^2(\text{BSO}(4), \mathbb{Z}_2)$ is the one-dimensional $\mathbb{Z}_2$ linear space with basis the class W_2. It therefore follows that there exists a number z in $\mathbb{Z}_2$ such that $\theta_2 = z \,.\, W_2$. Now $z = 0$ or 1. We demonstrate that $z = 1$ by constructing a compact orientable four-manifold X whose four-dimensional obstruction classes (P_1 and E, see below) are trivial but which is non-parallelisable with $W_2(X) \neq 0$. Let X_0 be the total space of the non-trivial two-sphere bundle over S^2. (The isomorphism classes of such bundles are in one-to-one correspondence with the set of isomorphism classes of associated principal SO(3) bundles. By universality, the latter set is just the set of homotopy classes of maps from S^2 into BSO(3), i.e. $\Pi_2(\text{BSO}(3)) \cong \Pi_1(\text{SO}(3)) \cong \mathbb{Z}_2$, implying that there is just one non-trivial class.) It can be shown that X_0 is diffeomorphic to the connected sum $\mathbb{C}P^2_+ \,\&\, \mathbb{C}P^2_-$ of two oppositely oriented copies of the complex projective plane $\mathbb{C}P^2_+$. Note that $W_2(X_0) \neq 0$ since by using methods similar to those used in the proof of part (1) of theorem 1, one can derive a class x in $H^2(X_0, \mathbb{Z}_2)$ with $x \cup x \neq 0$ from the Stiefel–Whitney class $W_2(\mathbb{C}P^2)$. The latter has $W_2(\mathbb{C}P^2) \cup W_2(\mathbb{C}P^2) \neq 0$ because for any four-manifold X, $W_2(X) \cup W_2(X) \,.\, \langle X\rangle = \text{E} \,.\, \langle X\rangle \,\text{mod}(2)$ and $\text{E} \,.\, \langle \mathbb{C}P^2\rangle = 3$. $P_1\langle X_0\rangle = 0$ follows from the fact that X_0 is cobordant to the disjoint sum $\mathbb{C}P^2_+ \cup \mathbb{C}P^2_-$ which is the boundary of the cylinder $\mathbb{C}P^2 \times D^1$. $\text{E} \,.\, \langle X_0\rangle = 2$ is easily derived from Mayer–Vietoris arguments. Therefore using theorem 1, the modified manifold X_{0s} has $W_2 \neq 0$, $\text{E} \,.\, \langle X_{0s}\rangle = 0$ and $P_1 \,.\, \langle X_{0s}\rangle = 0$. X_{0s} cannot be parallelisable since it has at least one non-trivial characteristic class.

Note that the above manifold is an example of a compact space–time which cannot admit a spinor structure. Another example can easily be derived from the complex projective plane which this time has a non-trivial Pontryagin number. Take $Y = \mathbb{C}P^2_+ \,\&\, \mathbb{C}P^2_+$. Then it is easy to show that $\text{E} \,.\, \langle Y\rangle = 2$. $P_1 \,.\, \langle Y\rangle = +6$ follows most easily from the observation that Y is oriented cobordant to the disjoint sum $\mathbb{C}P^2_+ \cup \mathbb{C}P^2_+$ which trivially has signature +2. Y_s is therefore a compact space–time (because $\text{E} \,.\, \langle Y_s\rangle = 0$) but cannot be a spinor manifold because $P_1 \,.\, \langle Y\rangle = 6$ contradicts Rohlin's theorem.

(3) $\theta_4 \in H^4(\text{BSO}(4), \mathbb{Z}\oplus\mathbb{Z})$. It is easy to show that the cohomology group is isomorphic to the direct sum of two copies of $H^4(\text{BSO}(4), \mathbb{Z})$. Recall that the latter group is generated by E and P_1 (modulus elements of order two, which we ignore since we are really interested in the images of θ_4 in groups $H^4(X, \mathbb{Z}\oplus\mathbb{Z})$ which are free Abelian). But then it follows that $H^4(\text{BSO}(4), \mathbb{Z}\oplus\mathbb{Z})$ is freely generated over the ring $\mathbb{Z}\oplus\mathbb{Z}$ by the universal Euler class E and the universal Pontryagin class P_1. We therefore have to find elements (a_1, a_2) and (b_1, b_2) of $\mathbb{Z}\oplus\mathbb{Z}$ such that $\theta_4 = (a_1, a_2) \,.\, \text{E} + (b_1, b_2) \,.\, P_1$. Without being able to compute these numbers explicitly, we can show that neither is trivial by geometrical constructions.

(*a*) $(a_1, a_2) \neq 0$. This is demonstrated by exhibiting a compact four-manifold X with

$W_2(X) = 0$, $P_1 . \langle X \rangle = 0$, but $E . \langle X \rangle \neq 0$. The obvious example is S^4 which has $W_i = 0$ for all $i \neq 0$, $P_1 . \langle S^4 \rangle = 0$ following from the fact that $S^4 = \partial_0 D^5$ and $E . \langle S^4 \rangle = 2$.

(*b*) $(b_1, b_2) \neq 0$. Suppose that we were able to find a starting manifold X with non-trivial Pontryagin number, trivial second Stiefel–Whitney class and with a positive even Euler number. Then, using theorem 1, we could obtain a modified manifold X_s with non-trivial Pontryagin number, trivial second Stiefel–Whitney class and a trivial Euler number. The existence of such a manifold would verify the assertion $(b_1, b_2) \neq 0$. Two types of starting manifold exist. The classic example is the Kummer surface (Spanier 1958). This space is derived from the four-torus $(S^1)^4$ under the singular $\mathbb{Z}_2$ action represented by complex conjugation in each factor of the product. The quotient space $(S^1)^4/\mathbb{Z}_2$ is a manifold except around the sixteen singular points $(\pm 1, \pm 1, \pm 1, \pm 1)$. The Kummer surface is obtained by 'smoothing out' these sixteen singular points. Its characteristic classes, derived in (Spanier 1958) are $P_1 . \langle X \rangle = 48$, $E . \langle X \rangle = 24$, and $W_2 = 0$. By theorem 1, twelve spherical modifications will kill the Euler number but leave $P_1 . \langle X \rangle = 48$ and $w_2(X) = 0$.

A whole family of possible starting manifolds is the set of non-singular algebraic surfaces in $\mathbb{C}P^3$ of degree four (Milnor 1958, Hirzebruch 1966) which have $E . \langle X \rangle = 24$, $P_1 . \langle X \rangle = -48$ and $W_2(X) = 0$.

Note that any of the above spherically modified compact four-manifolds provide direct counter examples to the assertion by Lee (1975), all being compact non-parallelisable spinor space–times.

We are now in a position to re-derive the theorem of Hirzebruch and Hopf.

Theorem 2. A smooth, compact, orientable four-manifold is parallelisable if and only if $P_1 = 0$, $E = 0$ and $W_2 = 0$.

Proof. Clearly, if P_1, E and W_2 are all trivial, all the obstructions must vanish. Conversely, if a four-manifold is parallelisable, all its characteristic classes must be trivial.

Corollary. A compact spinor space–time is parallelisable if and only if its Pontryagin number is trivial, or equivalently, if and only if it is a spin boundary.

Proof. The first assertion is obvious by Steenrod's theorem. The second follows from the fact that an oriented four-manifold is an oriented boundary if and only if its Pontryagin number is trivial. (An oriented boundary with a spin structure must therefore be spin cobordant to a spinor boundary and hence spin cobordant.)

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